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States on Operator Algebras and Axiomatic System of Quantum Theory

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Recent development brings new results on the interplay of states on operator algebras and axiomatics of quantum mechanics. Neither hidden space in the sense of Kochen and Specker nor approximate hidden variables exist on von Neumann algebras. Tracial properties of states are connected with dispersions. The axioms on composite systems simplify to state extension properties.

KEY WORDS: states on operator algebras; approximate hidden variables; tensor products and independence of C^*-algebras.

1. INTRODUCTION

This review paper is focused on the relationship of axiomatic foundations of quantum theory and theory of states on operator algebras. We comment on some aspects of the recent development in this interdisciplinary area that are interesting both from mathematical and physical standpoint.

The theory of operator algebras has its origin in creating a firm mathematical base of quantum mechanics in the pioneering works of von Neumann, Murray, Jordan, Segal, Stone, and others (von Neumann, 1995). Since then this part of functional analysis has been developing steadily and has not lost its connections with quantum theory. Many concepts developed formerly in a mathematical context have found important applications in physics and vice versa. A well-known example in this regard is the concept of a KMS state that describes equilibrium in quantum statistical mechanics and is simultaneously crucial for the structure theory of von Neumann algebras.

Recent investigations have brought many results on state spaces that shed new light on basic axioms of quantum theory. We are going to address two of them—the theory of hidden variables and the tensor product structures in quantum mechanics. The paper is organized as follows. In Section 2 we overview basic concepts of the theory of operator algebras. In Section 3 we deal with the hidden variable problem

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in the C^*-approach to quantum mechanics. In the concluding section we provide new results on tensor products of C^*-algebras that allow one to simplify the axiom on composing quantum systems.

2. OPERATOR ALGEBRAS AND AXIOMATICS OF QUANTUM THEORY

There are various approaches to the mathematical foundations of quantum theory (Araki, 1999; Gudder 1979; Mackey, 1963; Varadarajan, 1968). In all of them the basic role is played by the duality, $\langle \cdot, \cdot \rangle$, between the linear structure, O, of observables and the convex structure, S, of states. If $a \in O$ and $\varrho \in S$, then the value $\langle a, \varrho \rangle$ represents the expectation value of the observable a provided that the quantum system is prepared in the state ϱ. If a is a quantum proposition (i.e., an observable with two possible values 0 and 1), then the value $\langle a, \varrho \rangle$ amounts to the probability of detecting value 1 when the system is in the state ϱ. In C^*-algebraic quantum mechanics the system of observables is given by the self-adjoint part of a C^*-algebra A. The set of states is given by the set of norm one positive functionals on A. In particular, quantum propositions are identified with projections in Hilbert spaces. This idea goes back to Birkhoff and von Neumann (1936) and Mackey (1963).

Before formulating basic postulates of quantum mechanics, we recall a few standard notions of operator theory and fix the notation. For more details on operator algebras we refer the reader to standard monographs (Dixmier, 1977; Kadison and Ringrose, 1986; Pedersen, 1979; Takesaki, 1979). In the sequel, $B(H)$ will denote the set of all bounded operators acting on a Hilbert space H. Let us view $B(H)$ as a $*$-algebra, i.e., as a linear space with multiplication given by compositions of operators, and an involutive $*$-operation assigning to each operator its adjoint. $B(H)$ endowed with the usual operator norm becomes a complete normed $*$-algebra. In its concrete form, a C^*-algebra is defined as a norm closed $*$-subalgebra of $B(H)$. C^*-algebras can be characterized as normed $*$-algebras having certain properties. Indeed, in its abstract form, C^*-algebra A is a complete normed $*$-algebra such that the following conditions are fulfilled for all $a, b \in A$: (i) $\|ab\| \le \|a\| \cdot \|b\|$; (ii) $\|a^* a\|^2 = \|a\|^2$. We shall denote by A_{sa} the (real) subspace of A consisting of all self-adjoint elements in A. The bridge between Hilbert spaces and C^*-algebras is provided by states. A positive functional on a C^*-algebra A is a linear functional ϱ on A such that $\varrho(a^* a) \ge 0$ for all $a \in A$. A state on a C^*-algebra A is a norm one positive functional on A. By the symbol $S(A)$ we shall denote the convex set of all states on A. Pure states are defined as extreme points of the set $S(A)$. Given a state ϱ on a C^*-algebra A one can endow A with pseudoinner product $(a, b)_\varrho = \varrho(a^* b)$ $(a, b \in A)$. Then A can be represented on the Hilbert space H_ϱ obtained from $(A, (\cdot, \cdot)_\varrho)$ by factoring out zero vectors and completing. In particular, there is a Hilbert space H_ϱ, a unit vector $\xi_\varrho \in H_\varrho$, and a

$*$-homomorphism π_ϱ of A into $B(H_\varrho)$ such that $\varrho(a) = (\pi_\varrho(a)\,\xi_\varrho, \xi_\varrho)$ for all $a \in A$ and such that the set $\{\pi_\varrho(a)\,\xi_\varrho \mid a \in A\}$ is dense in H_ϱ. The representation π_ϱ is called the Gelfand–Naimark–Segal representation (in short G.N.S. representation) of ϱ and it is uniquely determined up to a unitary map. We say that a state ϱ is faithful if $(\cdot, \cdot)_\varrho$ is an inner product, i.e., if $\varrho(a^*\,a) > 0$ for all nonzero $a \in A$.

The C^*-algebras are simultaneous generalizations of finite-dimensional and noncommutative matrix algebras and commutative and infinite-dimensional algebras of continuous functions on compact spaces. All states on the algebra $M_n(\mathbb{C})$ of all complex n by n matrices are convex combinations of vector states. A state on a C^*-algebra A is called a vector state if it is of the form $\omega_\xi(a) = (a\,\xi, \xi)\ (a \in A)$, where ξ is a unit vector in the underlying Hilbert space. States on algebra $C(X)$ of all continuous functions on a compact Hausdorff space X are in one-to-one correspondence with Radon probability measures on X. A state ϱ on a C^*-algebra A is called tracial (or a trace) if $\varrho(a^*\,a) = \varrho(a\,a^*)$ for all $a \in A$. A state ϱ on the matrix algebra $M_n(\mathbb{C})$ is a trace if and only if $\varrho(a) = \frac{1}{n}\sum_{i=1}^{n}(a\,e_i, e_i)$, $(a \in M_n(\mathbb{C}))$, where (e_i) is (any) orthonormal basis of $\mathbb{C}^n$.

C^*-algebras represented on Hilbert spaces are closed with respect to the uniform convergence on the unit ball of the underlying Hilbert space. In general, they need not be closed with respect to the strong operator topology. We say that a net (a_α) of bounded operators on a Hilbert space H converges in the strong operator topology to an operator $a \in B(H)$ if $a_\alpha\,\xi$ converges to $a\,\xi$ in the norm topology on H for all $\xi \in H$. A C^*-algebra is called a von Neumann algebra if it is closed with respect to the strong operator topology. Von Neumann algebras are characterized as C^*-algebras which are dual Banach spaces. A projection in a C^*-algebra is an element p such that $p^2 = p^* = p$. It is well known that von Neumann algebras have rich projection structures. We shall denote by $P(M)$ the set of all projections in a von Neumann algebra M. Endowed with the order $p \le q$ if and only if $pq = p$, and with an orthocomplementation, $p^\perp = 1 - p$, the structure $P(M)$ becomes a prominent example of a complete orthomodular lattice. An important class of states on von Neumann algebras is that of normal states. A state ϱ on a von Neumann algebra M is called normal if it preserves bounded monotone nets of self-adjoint operators in M, or equivalently, if the restriction of ϱ to the projection lattice $P(M)$ is a completely additive measure. Any normal state is a σ-convex combination of vector states and corresponds to a density matrix. In this form it is more familiar to physicists.

Now we pass to basic concepts of the structure theory of von Neumann algebras. The center, $Z(M)$, of a von Neumann algebra M is defined by $Z(M) = \{x \in M \mid xy = yx \text{ for all } y \in M\}$. It is always an abelian von Neumann subalgebra of M. M is called a factor if the center of M consists only of multiples of the identity. Factors are considered to be building blocks of general von Neumann algebras. The algebra $B(H)$ is a factor. The direct sum $M_1 \oplus M_2$ of von Neumann algebras M_1 and M_2 is defined as the direct sum of linear spaces M_1 and M_2 with coordinatewise defined arithmetic operations and the norm $\|a \oplus b\| = \max\{\|a\|, \|b\|\}$ $(a \in M_1, b \in M_2)$. The direct sum of von Neumann algebra s is again a von Neumann algebra. (It can be verified that any central element $z \in Z(M)$ induces the direct sum decomposition $z\,M \oplus (1-z)\,M$ of M and vice versa.) We say that a von Neumann algebra A is the direct summand of a von Neumann algebra M if there is a von Neumann algebra B such that M can be written as $M = A \oplus B$. It can be proved that there is a largest abelian direct summand, M_{ab}, of M. The simplest noncommutative von Neumann algebra is the algebra $M_2(\mathbb{C})$ of all 2 by 2 matrices. This algebra constitutes an example of a type I_2 factor. In general case, we say that a von Neumann algebra is of type I_2 if it is $*$-isomorphic to the algebra $M_2(A)$ of all 2 by 2 matrices over an abelian von Neumann algebra A. Any von Neumann algebra can be written as a direct sum $M_1 \oplus M_2$, where M_1 is either zero or of type I_2, and M_2 is either zero or does not contain any type I_2 direct summand.

A C^*-algebra A is called a real rank zero algebra if any element in A_{sa} can be approximated by a linear combination of projections. Besides von Neumann algebras, the class of real rank zero algebras comprises many C^*-algebras relevant to physics such as CCR algebras, rotation algebras, and Cuntz algebras. A C^*-algebra A is called simple if every closed two-sided ideal in A is either zero or A.

Now we turn to the tensor products of C^*-algebras. Let L and K be linear spaces. By $L \otimes_{\mathrm{alg}} K$ we shall denote their algebraic tensor product. It is linearly generated by simple tensors, $l \otimes k$, where $l \in L$ and $k \in K$. The tensor product, $H \otimes K$, of Hilbert spaces H and K is the completion of the inner product space $H \otimes_{\mathrm{alg}} K$ with inner product given by the following natural rule on simple tensors: $(h_1 \otimes k_1, h_2 \otimes k_2) = (h_1, h_2) \cdot (k_1, k_2)$, $(h_1, h_2 \in H; k_1, k_2 \in K)$. Given now two operators $x \in B(H)$ and $y \in B(K)$, there is a unique bounded operator, $x \otimes y$, on $H \otimes K$ such that $(x \otimes y)(h \otimes k) = x\,h \otimes y\,k$ for all $h \in H$ and $k \in K$. Let A and B be C^*-algebras acting on H and K, respectively. The spatial tensor product, $A \otimes B$, of algebras A and B is a C^*-algebra which acts on $H \otimes K$ and is generated by the set $\{x \otimes y \mid x \in A, y \in B\}$. It should be remarked that the spatial tensor product (as an abstract C^*-algebra) does not depend on the faithful representations of A and B on concrete Hilbert spaces H and K, respectively.

The algebraic tensor product $A \otimes_{\mathrm{alg}} B$ can carry more than one C^*-norm. It can be shown that $A \otimes_{\mathrm{alg}} B$ becomes a $*$-algebra with multiplication and involution given by the following rules: $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1\,a_2 \otimes b_1\,b_2$, $(a_1 \otimes b_1)^* = a_1^* \otimes b_1^*$, $(a_1, a_2 \in A, b_1, b_2 \in B)$. Any C^*-norm β on $A \otimes_{\mathrm{alg}} B$ is defined as a norm on $A \otimes_{\mathrm{alg}} B$ such that the completion of $A \otimes_{\mathrm{alg}} B$ with respect to β is a C^*-algebra. The corresponding completion is denoted by $A \otimes_\beta B$ and it is called the (abstract) tensor product of A and B. It is a remarkable fact, due to Takesaki, that there is a C^*-norm, $\|\cdot\|_{\min}$, on $A \otimes_{\mathrm{alg}} B$ such that $\|\cdot\|_{\min} \le \beta$ for any C^*-norm β on $A \otimes_{\mathrm{alg}} B$. The norm $\|\cdot\|_{\min}$ is called the minimal norm. A deep result on tensor norms says that the minimal norm coincides with the C^*-norm of the spatial tensor product. As a consequence, the identity map on $A \otimes_{\text{alg}} B$ extends to a continuous homomorphism of $A \otimes_\beta B$ onto $A \otimes_{\min} B$. In other words, the minimal (i.e., spatial) tensor product is a quotient of any tensor product.

Besides tensoring algebras, one can also tensor their state spaces. It can be proved that for any pair of states ϱ and φ on C^*-algebras A and B, respectively, there is unique state, $\varrho \otimes \varphi$ on $A \otimes_\beta B$ such that $(\varrho \otimes \varphi)(a \otimes b) = \varrho(a)\,\varphi(b)$ for all $a \in A, b \in B$. The state $\varrho \otimes \varphi$ is called the tensor product of states ϱ and φ.

After having reviewed the basic concepts of the theory of C^*-algebras needed in the sequel, we now summarize the basic axioms of C^*-algebraic quantum mechanics:

- The set of all observables of a quantum system S is the self-adjoint part of a C^*-algebra A.
- The set of all states of a quantum system S is the state space, $S(A)$, of the C^*-algebra A.
- The value $\varrho(a)$, where $\varrho \in S(A)$ and $a \in A_{\text{sa}}$ is the expectation value of an observable a on the condition that a system S is prepared in the state ϱ.
- Evolution of a system S is given by a specified class of morphisms of the C^*-algebra A (unitary maps, automorphisms, completely positive maps).
- Given independent quantum systems S_1 and S_2 represented by C^*-algebras A and B, respectively, the smallest composite system containing S_1 and S_2 is given by the minimal tensor product $A \otimes_{\min} B$.

3. HIDDEN VARIABLES

In this section, results on hidden variables in C^*-quantum mechanics are surveyed. We show the link between approximate hidden variables and tracial properties of states and prove a new generalization of the classical Kochen–Specker Theorem.

A discussion of the problem of hidden variables is a part of the fundamental question on completeness of quantum mechanics. In general terms, the problem of the existence of hidden variable theory is the question of whether or not the probability interpretation of quantum mechanics can be eliminated by constructing a probability space such that the probability of any quantum event can be described by a classical random variable. Results obtained so far indicate that it is not possible to do it for all observables simultaneously. We are going to show some new evidence in this respect.

Let L be an orthomodular lattice. By a *dispersion-free state* on L we mean a finitely additive probability measure on L with values in the set $\{0, 1\}$. Any probability measure on a probability space can be written as an "integral" mixture of dispersion-free states, e.g. as a statistical mixture of Dirac measures concentrated at points of the underlying "phase space." We shall show that this does not hold in the operator algebraic approach to quantum theory, the main reason being the nonexistence of a dispersion-free state at all. The first result in this direction is a well known von Neumann impossibility proof (Caruana, 1995) stating that there is no normal state on $B(H)$, where $\dim H = \infty$, such that its restriction to the projection lattice of $B(H)$ is dispersion-free. This no-go result was extended by Plymen, 1968, who proved that there is no dispersion-free normal state on a von Neumann algebra M without a one-dimensional direct summand. Similar conclusions for complete orthomodular lattices have been obtained by Jauch and Piron, 1963, 1969. The following theorem is the strongest no-go result for dispersion-free states in the context of von Neumann algebras.

Theorem 3.1. (Hamhalter, 1993) *The projection lattice $P(M)$ of a von Neumann algebra M which has neither a nonzero Abelian nor a type I_2 direct summand admits no dispersion free state.*

This result can be proved by using Gleason's Theorem for positive measures on von Neumann algebras (Aaarnes, 1969; Aaarnes, 1970; Christensen, 1982; Gleason, 1957; Hamhalter, 2003; Maeda, 1990; Yeadon, 1983, 1984) and then reduced to the problem of the existence of a linear multiplicative state on a von Neumann algebra. However, since the proof of Gleason's Theorem is very difficult, it is desirable to use more elementary arguments. On employing the following proposition we see that the problem of the existence of a dispersion-free state comes down to simple matrix algebras.

Proposition 3.2. *Let M be a von Neumann algebra with no nonzero abelian direct summand and no type I_2 direct summand. The following statements hold:*

(i) Any subalgebra of M which is $*$-isomorphic to $M_2(\mathbb{C})$ is contained in a subalgebra $C \oplus D$ of M satisfying the following properties: C is either zero or it is $*$-isomorphic to $M_4(\mathbb{C})$; D is either zero or it is is a copy of $M_2(\mathbb{C})$ contained in another subalgebra of M which is $*$-isomorphic to $M_3(\mathbb{C})$.

(ii) M contains a unital subalgebra $*$-isomorphic to one of the following matrix algebras: $M_2(\mathbb{C})$, $M_3(\mathbb{C})$, $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$.

Statement (i) above has been proved in the course of the proof of Gleason's Theorem (Christensen, 1982; Yeadon, 1983,1984) and it is summarized in (Hamhalter, 2003, Proposition 5.3.6, p. 132). Statement (ii) has been proved in (Theorem 7.3.1, Hamhalter, 2003).

The previous proposition implies that it is enough to prove the nonexistence of dispersion-free states on algebras $M_3(\mathbb{C})$ and $M_4(\mathbb{C})$. In these cases more elementary arguments based on spherical geometry can be used instead of applying Gleason's Theorem in its full power. Various simplifications in this regard have been obtained by Piron (1968), Bell (1966), Navara (2004), and Hamhalter (2003, Theorem 3.4.1).

One of the consequences of Theorem 3.1 is that the projection lattices of nearly all von Neumann algebras cannot be embedded into Boolean algebras in such a way that the embeddings respect suprema of finite collections of orthogonal projections. This fact is connected with the non-existence of the hidden space in the sense of Kochen and Specker, 1967. A *Hidden space* of a given quantum system is a set, Ω, with a σ-field, $\mathcal{A}$, of subsets of Ω with the following properties: For each quantum observable A and for each quantum state ϱ there are an $\mathcal{A}$-measurable function $f_A : \Omega \to \mathbb{R}$ and a probability measure μ_ϱ on $\mathcal{A}$, respectively, such that the following conditions are fulfilled:

(i) For each Borel set $\mathcal{B} \subset \mathbb{R}$ the probability that the value of an observable A is in $\mathcal{B}$ equals $\mu_\varrho(f_A^{-1}(\mathcal{B}))$, provided that the system is in the state ϱ.
(ii) (Function Principle) If A and B are observables such that $B = g(A)$, where g is a real Borel function, then $f_B = g \circ f_A$.

The first condition says that any observable is statistically equivalent to a certain random variable on Ω. Condition (ii) means that the representation by random variables preserves transformation rules for observables. For a more detailed discussion on this topic see (Döring, 2004). The famous Kochen–Specker Theorem (Kocher and Specker, 1967) tells us that the quantum model given by the algebra $B(H)$, where H is a separable Hilbert space of dimension at least 3, has no hidden space Ω. Recently, it has been proved by Döring, 2004, that a hidden space does not exist for any von Neumann algebra without a type I_2 and a nonzero Abelian direct summand that acts on a separable Hilbert space. By a different method we generalize this result to all von Neumann algebras without abelian and a type I_2 direct summand. Moreover, it turns out that only the validity of the Function Principle suffices for excluding a hidden space. No reference to the set of states is needed.

Theorem 3.3. *Let M be a von Neumann algebra without a type I_2 direct summand and with no nonzero abelian direct summand. There is no σ-field $(\Omega, \mathcal{A})$ and a map $a \to f_a$ assigning to each self-adjoint element a in M an $\mathcal{A}$-measurable real function f_a on Ω such that $f_{g(a)} = g \circ f_a$ for any real continuous function g on $\mathbb{R}$.*

Proof: The idea of the proof is to show that the existence of such a σ-field $(\Omega, \mathcal{A})$ implies the existence of a dispersion-free probability measure on $P(M)$. Suppose that $(\Omega, \mathcal{A})$ has the properties stated in the theorem. Fix $\omega \in \Omega$ and consider the map $s : M_{sa} \to \mathbb{R}$, $s(a) = f_a(\omega)$. If $p \in P(M)$, then $p^2 = p$ and so $f_p(\omega)^2 = f_p(\omega)$. In other words, $s(p) \in \{0, 1\}$. Let us now take orthogonal nonzero projections $p, q \in M$. Put $x = p + \frac{1}{2}q$. Let f and g be continuous real functions on $\mathbb{R}$ such that $f(1) = g(\frac{1}{2}) = 1$ and $f(\frac{1}{2}) = g(1) = 0$. It is clear that $f(x) = p$ and $g(x) = q$. Using the Function Principle we obtain

$$s(p+q) = s(f(x) + g(x)) = s((f+g)(x)) = (f+g)(f_x(\omega))$$
$$= s(f(x)) + s(g(x)) = s(p) + s(q)\,.$$

Therefore, s is a finitely additive dispersion-free measure on $P(M)$. On employing the Function Principle to a unit element, $\mathbf{1}$, of M and constant unit function on $\mathbb{R}$, we see that $f_{\mathbf{1}}(\omega) = 1$ for all $\omega \in \Omega$. Hence, s induces a dispersion-free state on $P(M)$, which is in contradiction with Theorem 3.1. □

The problem of the existence of dispersion-free states and related questions on hidden variables for von Neumann algebra are more or less solved. The situation for C^*-algebras is not so clear. The reason is the lack of Gleason type theorems. If ϱ is a state on a C^*-algebra A and the linear span of projections in A is dense in A, then ϱ induces a dispersion-free state on the projection structure of A if and only if ϱ is a $*$-homomorphism of A into the complex field. Therefore, A has a linear dispersion-free state if and only if it contains a closed ideal of codimension 1. For C^*-algebras with small or empty projection structures, probability measures on projections are replaced by quasi-states. A function $\varrho : A \to \mathbb{C}$ on a C^*-algebra A is called *positive quasi-functional* if (i) ϱ is a positive linear functional on all abelian C^*-subalgebras of A, (ii) $\varrho(a + i\,b) = \varrho(a) + i\,\varrho(b)$ for all self-adjoint $a, b \in A$. The norm of ϱ is defined as $\|\varrho\| = \sup\{|\varrho(a)| \mid \|a\| \leq 1\}$. If $\|\varrho\| = 1$ we call ϱ a *quasi-state*. The quasi-functional ϱ is called *monotone* if $\varrho(a) \leq \varrho(b)$ whenever $a \leq b$. It is not known whether a quasi-state on a C^*-algebra not having a quotient $*$-isomorphic to $M_2(\mathbb{C})$ is linear. However, quasi-functionals seem to be more natural for the axiomatics of quantum theory because they postulate additivity only with respect to commuting elements. Let us denote by $Q(A)$ the set of all positive quasi-functionals of norm less then one. Denote further by $M(A)$ the set of all monotone multiplicative functionals in $Q(A)$. (A quasi-functional ϱ is called *multiplicative* if $\varrho(a\,b) = \varrho(a)\,\varrho(b)$ for all $a, b \in A$). The following result of Misra states that non-Abelian C^*-algebras do not have large sets of multiplicative monotone quasi-functionals.

Theorem 3.4. (Misra, 1967). *A C^*-algebra A is Abelian if and only if the closure of the convex hull of $M(A)$ in the topology of pointwise convergence on elements of A is $Q(A)$.*

However, it is not clear in what cases the set $M(A)$ is empty. We succeeded in proving the following C^*-version of von the Neumann impossibility proof.

Theorem 3.5. (Hamhalter, 2004) *Let A be a simple infinite unital C^*-algebra. Then A does not admit any dispersion-free quasi-state.*

The existence of hidden variables has to be verified by an experiment. Since every real measurement admits an error, it seems to be more realistic to formulate the question on the existence of hidden variables as the problem of the existence of a state with arbitrarily small dispersion. The problem of approximate hidden variables was posed by G.W. Mackey (see Jauch, 1968). In this connection a quantitative measure of dispersion, called the overall dispersion, has been introduced. Let ϱ be a state on a projection structure $P(A)$ of a C^*-algebra A. The *overall dispersion*, $\sigma(\varrho)$, of ϱ is defined by

$$\sigma(\varrho) = \sup\{\varrho(p) - [\varrho(p)]^2 \mid p \in P(A)\}\,.$$

Any overall dispersion is a number between 0 and 1/4. The dispersion $\sigma(\varrho) = 0$ if and only if ϱ is dispersion-free. Study of possible values of dispersions brings interesting observations even for matrix algebras. It was proved in (Hamhalter, 2004) that any state on a matrix algebra of even rank has dispersion 1/4, while on algebras of odd ranks the trace is the only state with a smallest dispersion. This simple but useful fact demonstrates the difference between classical and quantum theory. The random variable on a discrete finite probability space has a smallest dispersion if and only if it is a Dirac measure. On the other hand, a discrete quantum observable with an odd number of values has minimal dispersion if and only if it is a trace, i.e., a uniform mixture of pure states. As a consequence, we see that the dispersion of states on matrix algebras is uniformly bounded from below. This fact can be generalized to all C^*-algebras with reasonably many projections.

Theorem 3.6. (Hamhalter, 2004) *Let A be a unital real rank zero algebra having no representation onto an abelian C^*-algebra. Then*

$$\sigma(\varrho) \geq \frac{2}{9}\,,$$

for any state ϱ on A.

This theorem excludes the existence of states with arbitrarily small dispersions on the projection lattices of von Neumann algebras not having a type I_2 and a nonzero abelian direct summand. (The exclusion of the type I_2 and the abelian part is necessary by Hamhalter, 1993.) Normalized traces on matrix algebras of odd ranks can be characterized as the states with the smallest dispersions. In the following theorem we generalize this fact to all von Neumann algebras.

Theorem 3.7. (Hamhalter, 2004) *Let M be a von Neumann algebra with no nonzero Abelian direct summand and no Type I_2 direct summand. A state φ on*

M is tracial if and only if the following condition holds: For any von Neumann subalgebra A of M which is $*$-isomorphic to $M_3(\mathbb{C})$ and such that the restriction, ϱ_A, of ϱ to A is nonzero, the state $\varrho_A/\|\varrho_A\|$ has a smallest dispersion among all states on A.

Traces play a central role in the classification of von Neumann algebras and are characterized from many points of view. The previous characterization of traces in terms of dispersions is a new one. Let us remark that this characterization can also be extended to weights on von Neumann algebras (Hamhalter, 2004).

4. PRODUCT STATES AND TENSOR STRUCTURE

In the concluding part of this paper we investigate the relationship between independence of operator algebras, product properties of states, and tensor structures. All the result presented here have been obtained jointly by L. J. Bunce and the author in Bunce and Hamhalter (2004). We characterize tensor products of C^*-algebras in quite simple terms of state extensions and, in particular, without assuming their mutual commutation. This clarifies the position of the spatial tensor product in the axiomatics of quantum theory.

In the sequel let A and B be C^*-subalgebras of a unital C^*-algebra D such that $1 \in A, B$. If $D = A \otimes_\beta B$, then A and B will be identified with $A \otimes 1$ and $1 \otimes B$, respectively. This tensor product organization represents a special configuration of the given algebras. For instance, the tensor norm is always a cross norm, i.e., $\|a \otimes b\| = \|a\| \cdot \|b\|$ $(a \in A, b \in B)$. Further, each pair of states on algebras A and B has unique common extension to a product state on D. The tensor product is a model of independent quantum mechanical systems. It is adopted for the reason of mathematical convenience and cannot be used so directly in more general quantum theories such as quantum field theory (Hamhalter, 1997, 2002, 2003; Summers, 1990). For this reason many other independence conditions have been studied. Haag and Kastler (1964) proposed the following major concept of independence. C^*-algebras A and B are called *C^*-independent* if for each state ϱ on A and for each state τ on B there is a unique state φ on D extending both ϱ and τ. The C^*-independence has a direct physical meaning: Each system can be prepared in an arbitrary state without disturbing the other. The C^*-independence does not imply mutual commutation of the algebras. Therefore, the following interesting question, known as the commutation problem, arises: Under what additional conditions does the C^*-independence of A and B imply that (i) A and B commute? (ii) D is $*$-isomorphic to some tensor product of A and B? (iii) D is $*$-isomorphic to the spatial tensor product of A and B? If one supposes that A and B commute, then the tensor product really emerges. This is the content of the classical Roos' Theorem in quantum field theory.

Theorem 4.1. (Roos, 1974) *Assume that A and B commute. The following conditions are equivalent:*

(i) A and B are C^*-indepedent.
(ii) D is $*$-isomorphic to $A \otimes_\beta B$ for some C^*-norm β.
(iii) For each state ϱ on A and for each state τ on B there is a unique state φ on D such that $\varphi(a\,b) = \varrho(a)\,\tau(b)$ for all $a \in A$, $b \in B$.

The proof of Roos' Theorem uses in an essential way the fact that A and B commute. In order to generalize this result to noncommuting algebras we have introduced the following concepts. A state φ on D is said to be a *product state across* A and B if $\varphi(a\,b) = \varphi(a)\,\varphi(b)$ for all $a \in A$ and $b \in B$. A state φ is said to be an *uncoupled product state across* A and B if

$$\varphi(a_1\,b_1\,a_2\,b_2 \cdots a_n\,b_n) = \varphi(a_1\,a_2 \cdots a_n) \cdot \varphi(b_1\,b_2 \cdots b_n)$$

for all $a_1, \dots, a_n \in A, b_1, \dots, b_n \in B$. Every uncoupled product state is a product state; the reverse implication is not valid. If A and B commute, then both concepts coincide. We say that D has the *C^*-uncoupled product property across A and B* if for every pair of states $\varrho \in S(A)$ and $\tau \in S(B)$ there is an uncoupled product state φ on D extending both ϱ and τ. The following result gives Roos' Theorem if A and B commute.

Theorem 4.2. (Bunce and Hamhalter, 2004) *The following conditions are equivalent:*

(i) D has the C^*-uncoupled product property across A and B.
(ii) There exists a unique C^*-norm β on $A \otimes_{\text{alg}} B$ and a $*$-isomorphism of $A \otimes_\beta B$ onto $D/J(A, B)$, sending canonically $a \otimes b$ to $a\,b + J(A, B)$ $(a \in A, b \in B)$, where $J(A, B)$ is the two-sided closed ideal generated by the set of all commutators $\{a\,b - b\,a \mid a \in A, b \in B\}$.

If D is a simple algebra, then $J(A, B)$ has to be zero and so A and B commute. Another contribution to the commutation problem is the following theorem:

Theorem 4.3. (Bunce and Hamhalter, 2004) *D is canonically $*$-isomorphic to the minimal tensor product $A \otimes_{min} B$ if and only if there is a set S of uncoupled product states across A and B such that the set of the corresponding G.N.S. representations $\{\pi_\varphi \mid \varphi \in S\}$ is faithful on D. (We say that a system of representations is faithful if the intersection of their kernels is zero.)*

One of the main results of our analysis is the following corollary:

Corollary 4.4. (Bunce and Hamhalter, 2004) *If D is simple and there is at least one uncoupled product state across A and B, then D is canonically $*$-isomorphic to $A \otimes_{\min} B$.*

Now we shall deal with the uniqueness of common state extensions. It turns out, surprisingly, that the set of all uncoupled product states is the only reasonable set in which unique extensions of states can be realized.

Theorem 4.5. (Bunce and Hamhalter, 2004) *Let Δ be a weak* closed subset of $S(D)$ satisfying the following conditions:*

(i) For each $\varrho \in S(A)$ and for each $\tau \in S(B)$ there is a unique state $\varphi \in \Delta$ extending both ϱ and τ.
(ii) For each $\varrho \in S(A)$ and each $\tau \in S(B)$, the sets $\{\varphi \in \Delta \mid \varphi|A = \varrho\}$ and $\{\varphi \in \Delta \mid \varphi|B = \tau\}$ are convex.

Then Δ is the set of all uncoupled product states across A and B.

The previous theorem says that under mild convexity conditions the set of unique extensions is nothing but the set of uncoupled product states. Uniqueness of states implies strong algebraic properties. One example of the set Δ satisfying assumptions of Theorem 4.5 is the set $S_\Pi(D)$ of all "partial product" states φ on D such that $\varphi(ab) = \varphi(a)\,\varphi(b)$ whenever $a \in A_{\mathrm{sa}}, b \in B_{\mathrm{sa}}$ and $ab = ba$. These states act as product probability measures on the probability spaces given by abelian subalgebras generated by simultaneously measurable observables $a \in A$ and $b \in B$. Combining our previous results we obtain the following characterizations of the minimal tensor products.

Theorem 4.6. (Bunce and Hamhalter, 2004) *Suppose that for each pair of states ϱ and τ of A and B, respectively, there is a unique state $\varphi \in S_\Pi(D)$ extending ϱ and τ. Let D have a faithful family of G.N.S. representations associated with $S_\Pi(D)$. Then D is canonically $*$-isomorphic to $A \otimes_{\min} B$.*

Corollary 4.7. (Bunce and Hamhalter, 2004) *Let D be simple. Then D is canonically $*$-isomorphic to the minimal tensor product $A \otimes_{\min} B$ if and only if for each pair of states ϱ and τ of A and B, respectively, there is a unique state in $S_\Pi(D)$ extending ϱ and τ.*

We believe that the stated results contribute to the simplification of the axioms on composite systems. Let us remark that one of the consequences of these axioms is quantum entanglement. This crucial quantum phenomenon is now frequently used in quantum information theory (Nielsen and Chuang, 2000). In case of simple algebras one is necessarily led to accepting the minimal tensor product if the following natural requirement is satisfied: Every pair of states on local algebras has a unique product extension. Such a lucid characterization of the simple spatial product is not apparent from the original definition.

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REFERENCES

Aarnes, J. F. (1969). Physical states on a C^*-algebra. *Acta Mathematica* **122**, 161–172.

Aarnes, J. F. (1970). Quasi-states on C^*-algebras. *Transactions of American Mathematical Society* **149**, 601–605.

Araki, H. (1999). Mathematical Theory of Quantum Fields, Oxford University Press, Oxford.

Bell, J. S. (1966). On the problem of hidden variables in quantum mechanics. *Review of Modern Physics* **XXXVIII**, 447–475.

Birkhoff, G. and von Neumann, J. (1936). The logic of quantum mechanics. *Annals of Mathematics* **37**, 823–843.

Bunce, L. J. and Hamhalter, J. (2004). C^*-Independence, Product States, and Commutation. *Annales de l'Institut Henri Poicaré* **5**(5), 1981–1995.

Caruana, L. (1995). John von Neumann's Impossibility Proof in historical perspective. *Physis–Rivista Internazionale di Storia della Scienza. Nupva Serie* **32**(1), 109–124.

Christensen, E. (1982). Measures on projections and physical states. *Communications in Mathematical Physics* **86**, 529–538.

Dixmier, J. (1997). *C^*-algebras*, North Holland, Amsterdam.

Döring, A. (2004). Kochen–Specker Theorem for von Neumann algebras. arXiv:quant-ph/048106v1, 16 Aug 2004.

Florig, M. and Summers, S. J. (1997). On the statistical independence of algebras of observables. *Journal of Mathematical Physics* **38**, 1318–1328.

Gleason, A. M. (1957). Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics* **6**, 885–893.

Gudder, S. P. (1979). Stochastic Methods in Quantum Mechanics, Elsevier, Amsterdam.

Haag, R. and Kastler, D. (1964). An algebraic approach to quantum field theory. *Journal of Mathematical Physics* **5**(7), 848–861.

Hamhalter, J. (1993). Pure Jauch–Piron states on von Neumann algebras. *Annales de l'Institut Henri Poincaré* **58**, 173–187.

Hamhalter, J. (1997). Statistical independence of operator algebras. *Annales de l'Institut Henri Poincaré* **27**(4), 447–462.

Hamhalter, J. (2002). C^*-independence and W^*-independence of von Neumann algebras. *Mathematische Nachrichten* **239–240**, 146–156.

Hamhalter, J. (2003). *Quantum Measure Theory, Vol. 134: Fundamental Theories of Physics*, Kluwer Academic, Dordrecht.

Hamhalter, J. (2004). Traces, dispersions and hidden variables. *Foundations of Physics Letters* **17**(6), 581–597.

Jauch, M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, MA.

Jauch, M. and Piron, C. (1963). Can hidden variables be excluded in quantum mechanics? *Helvetia Physica Acta* **36**, 827–837.

Jauch, M. and Piron, C. (1969). On the structure of quantum proposition systems. *Helvetia Physica Acta* **42**, 827–837.

Kadison, R. V. and Ringrose, J. R. (1986). *Fundamentals of the Theory of Operator Algebras*, Vols. I–IV, Academic Press, New York.

Kochen, S. and Specker, E. P. (1967). The problem of hidden variables in quantum mechanics. *Journal of Mathematical Mechanics* **17**, 59–87.

Maeda, S. (1990). Probability measures on projections in von Neumann algebras. *Review of Mathematical Physics* **1**, 235–290.

Mackey, G. W. (1963). *The Mathematical Foundations of Quantum Mechanics*, Benjamin, New York.

Misra, B. (1967). When can hidden variables be excluded in quantum mechanics? *Nuovo Cimento A* **47**, 841–859.

von Neumann, J. (1995). *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, NJ.

Navara, M. (2004). Geometrical lemmas related to Gleason's Theorem. *International Journal of Theoretical Physics* **43**, (2004), No. 7, to appear.

Nielsen, A. M. and Chuang, I. L. (2000). *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, UK.

Pedersen, G. K. (1979). *C^*-algebras and their Automorphisms Groups*, Academic Press, New York.

Piron, C. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, MA.

Plymen, R. J. (1968). *Dispersion free normal states* II, *Nuovo Cimento A* **LIV**(4), 862–870.

Roos, H. (1974). Independence of local algebras in quantum field theory. *Communications in Mathematical Physics* **37**, 273–286.

Summers, S. J. (1990). On the independence of local algebras in quantum field theory. *Review of Mathematical Physics* **2**(2), 201–247.

Takesaki, M. (1979). *Theory of Operator Algebras I*, Springer, Berlin.

Varadarajan, V. S. (1968). *Geometry of Quantum Theory*, Vol. I, Van Nostrand, Princeton, NJ.

Yeadon, F. J. (1983). Measures on projections in W^*-algebras of Type II_1. *Bulletin of London Mathematical Society* **15**, 139–145.

Yeadon, F. J. (1984). Finitely additive measures on projections in finite W^*-algebras. *Bulletin of London Mathematical Society* **16**, 145–150.